

# SECOND MAIN THEOREM AND UNICITY OF MEROMORPHIC MAPPINGS FOR HYPERSURFACES IN PROJECTIVE VARIETIES

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**ABSTRACT.** Let  $V$  be a projective subvariety of  $\mathbb{P}^n(\mathbb{C})$ . A family of hypersurfaces  $\{Q_i\}_{i=1}^q$  in  $\mathbb{P}^n(\mathbb{C})$  is said to be in  $N$ -subgeneral position with respect to  $V$  if for any  $1 \leq i_1 < \dots < i_{N+1} \leq q$ ,  $V \cap (\bigcap_{j=1}^{N+1} Q_{i_j}) = \emptyset$ . In this paper, we will prove a second main theorem for meromorphic mappings of  $\mathbb{C}^m$  into  $V$  intersecting hypersurfaces in subgeneral position with truncated counting functions. As an application of the above theorem, we give a uniqueness theorem for meromorphic mappings of  $\mathbb{C}^m$  into  $V$  sharing a few hypersurfaces without counting multiplicity. In particular, we extend the uniqueness theorem for linear nondegenerate meromorphic mappings of  $\mathbb{C}^m$  into  $\mathbb{P}^n(\mathbb{C})$  sharing  $2n + 3$  hyperplanes in general position to the case where the mappings may be linear degenerate.

## 1. INTRODUCTION AND MAIN RESULTS

This article is a continuation of our studies in [2]. To formulate the main result in [2], we recall the following.

Let  $N \geq n$  and  $q \geq N + 1$ . Let  $D_1, \dots, D_q$  be hypersurfaces in  $\mathbb{P}^n(\mathbb{C})$ . The hypersurfaces  $D_1, \dots, D_q$  are said to be in  $N$ -subgeneral position in  $\mathbb{P}^n(\mathbb{C})$  if  $D_{j_0} \cap \dots \cap D_{j_N} = \emptyset$  for every  $1 \leq j_0 < \dots < j_N \leq q$ .

Throughout this paper, sometimes we will identify a hypersurface in  $\mathbb{P}^n(\mathbb{C})$  with one of its defining homogeneous polynomials if there is no confusion. In [2], the authors proved the following result.

**Theorem 1.** *Let  $f$  be an algebraically nondegenerate meromorphic mapping of  $\mathbb{C}^m$  into  $\mathbb{P}^n(\mathbb{C})$ . Let  $\{Q_i\}_{i=1}^q$  be hypersurfaces of  $\mathbb{P}^n(\mathbb{C})$  in  $N$ -subgeneral position with  $\deg Q_i = d_i$  ( $1 \leq i \leq q$ ). Let  $d = \text{lcm}(d_1, \dots, d_q)$  and  $M = \binom{n+d}{n} - 1$ . Assume that  $q > \frac{(M+1)(2N-n+1)}{n+1}$ . Then, we have*

$$\left\| \left( q - \frac{(M+1)(2N-n+1)}{n+1} \right) T_f(r) \leq \sum_{i=1}^q \frac{1}{d_i} N_{Q_i(f)}^{[M]}(r) + o(T_f(r)) \right\|$$

The first aim of this article is to generalize the above Second Main Theorem to meromorphic mappings into projective varieties sharing hypersurfaces in subgeneral position.

We now give the following.

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*Definition 2.* Let  $V$  be a complex projective subvariety of  $\mathbb{P}^n(\mathbb{C})$  of dimension  $k$  ( $k \leq n$ ). Let  $Q_1, \dots, Q_q$  ( $q \geq k+1$ ) be  $q$  hypersurfaces in  $\mathbb{P}^n(\mathbb{C})$ . The family of hypersurfaces  $\{Q_i\}_{i=1}^q$  is said to be in  $N$ -subgeneral position with respect to  $V$  if for any  $1 \leq i_1 < \dots < i_{N+1} \leq q$ ,

$$V \cap \left( \bigcap_{j=1}^{N+1} Q_{i_j} \right) = \emptyset.$$

If  $\{D_i\}_{i=1}^q$  is in  $n$ -subgeneral position then we say that it is in *general position* with respect to  $V$ .

Now, let  $V$  be a complex projective subvariety of  $\mathbb{P}^n(\mathbb{C})$  of dimension  $k$  ( $k \leq n$ ). Let  $d$  be a positive integer. We denote by  $I(V)$  the ideal of homogeneous polynomials in  $\mathbb{C}[x_0, \dots, x_n]$  defining  $V$  and by  $H_d$  the  $\mathbb{C}$ -vector space of all homogeneous polynomials in  $\mathbb{C}[x_0, \dots, x_n]$  of degree  $d$ . Define

$$I_d(V) := \frac{H_d}{I(V) \cap H_d} \text{ and } H_V(d) := \dim I_d(V).$$

Then  $H_V(d)$  is called the Hilbert function of  $V$ . Each element of  $I_d(V)$  which is an equivalent class of an element  $Q \in H_d$ , will be denoted by  $[Q]$ ,

*Definition 3.* Let  $f : \mathbb{C}^m \rightarrow V$  be a meromorphic mapping. We say that  $f$  is degenerate over  $I_d(V)$  if there is  $[Q] \in I_d(V) \setminus \{0\}$  such that  $Q(f) \equiv 0$ . Otherwise, we say that  $f$  is nondegenerate over  $I_d(V)$ . It is clear that if  $f$  is algebraically nondegenerate, then  $f$  is nondegenerate over  $I_d(V)$  for every  $d \geq 1$ .

Our main theorem is stated as follows.

**Theorem 4.** Let  $V$  be a complex projective subvariety of  $\mathbb{P}^n(\mathbb{C})$  of dimension  $k$  ( $k \leq n$ ). Let  $\{Q_i\}_{i=1}^q$  be hypersurfaces of  $\mathbb{P}^n(\mathbb{C})$  in  $N$ -subgeneral position with respect to  $V$  with  $\deg Q_i = d_i$  ( $1 \leq i \leq q$ ). Let  $d$  be the least common multiple of  $d_i$ 's, i.e.,  $d = \text{lcm}(d_1, \dots, d_q)$ . Let  $f$  be a meromorphic mapping of  $\mathbb{C}^m$  into  $V$  such that  $f$  is nondegenerate over  $I_d(V)$ .

Assume that  $q > \frac{(2N - k + 1)H_V(d)}{k + 1}$ . Then, we have

$$\left\| \left( q - \frac{(2N - k + 1)H_V(d)}{k + 1} \right) T_f(r) \leq \sum_{i=1}^q \frac{1}{d_i} N_{Q_i(f)}^{[H_V(d)-1]}(r) + o(T_f(r)). \right.$$

We note that, the second main theorem for algebraically nondegenerate meromorphic mappings into projective subvarieties was firstly given by Min Ru [12] in 2004. In his result the family of hypersurfaces is assumed in general position and there is no truncation level for the counting functions, but the total defect is  $n + 1$ , which is the sharp number.

**Remark:**

(i) In the case where  $V$  is a linear space of dimension  $k$  and each  $H_i$  is a hyperplane, i.e.,  $d_i = 1$  ( $1 \leq i \leq q$ ), then  $H_V(d) = k + 1$  and Theorem 4 gives us the classical Second Main Theorem of Cartan-Nochka (see [8] and [9]).

(ii) It is easy to see that  $H_V(d) - 1 \leq \binom{n+d}{n} - 1$ . Furthermore, the truncated level  $(H_V(d) - 1)$  of the counting function in Theorem 4 is much smaller than the previous results of all other authors (cf. [1], [4]).

(iii) By a direct computation from Theorem 4, it is easy to see that the total defect is  $\frac{(2N - k + 1)H_V(d)}{k + 1}$ . Unfortunately, this defect is  $\geq n + 1$ .

(iv) Also the above notion of  $N$ -subgeneral position is a natural generalization from the case of hyperplanes. Therefore, in order to prove Theorem 4, we give a generalization of Nochka weights for hypersurfaces in complex projective varieties.

(v) From Cartan-Nochka's theorem, we may obtain a second main theorem by using Veronese embedding which embeds  $\mathbb{P}^n(\mathbb{C})$  into  $\mathbb{P}^{\binom{n+d}{n}-1}(\mathbb{C})$ . But in that case we need the condition that the family of hyperplanes corresponding to the initial family of hypersurfaces is still in subgeneral position in  $\mathbb{P}^{\binom{n+d}{n}-1}(\mathbb{C})$ , which is not satisfied if  $N < \binom{n+d}{n}$ .

As an application of Theorem 4, the second aim of this article is to give a uniqueness theorem for meromorphic mappings of  $\mathbb{C}^m$  into  $V$  sharing a few hypersurfaces without counting multiplicity.

**Theorem 5.** *Let  $V$  be a complex projective subvariety of  $\mathbb{P}^n(\mathbb{C})$  of dimension  $k$  ( $k \leq n$ ). Let  $\{Q_i\}_{i=1}^q$  be hypersurfaces in  $\mathbb{P}^n(\mathbb{C})$  in  $N$ -subgeneral position with respect to  $V$  and  $\deg Q_i = d_i$  ( $1 \leq i \leq q$ ). Let  $d$  be the least common multiple of  $d_i$ 's, i.e.,  $d = \text{lcm}(d_1, \dots, d_q)$ . Let  $f$  and  $g$  be meromorphic mappings of  $\mathbb{C}^m$  into  $V$  which are nondegenerate over  $I_d(V)$ . Assume that*

- (i)  $\dim(\text{Zero}Q_i(f) \cap \text{Zero}Q_i(g)) \leq m - 2$  for every  $1 \leq i < j \leq q$ ,
- (ii)  $f = g$  on  $\bigcup_{i=1}^q (\text{Zero}Q_i(f) \cup \text{Zero}Q_i(g))$ .

Then the following assertions hold:

- a) If  $q > \frac{2(H_V(d) - 1)}{d} + \frac{(2N - k + 1)H_V(d)}{k + 1}$ , then  $f = g$ .
- b) If  $q > \frac{2(2N - k + 1)H_V(d)}{k + 1}$ , then there exist  $N + 1$  hypersurfaces  $Q_{i_0}, \dots, Q_{i_N}$ ,  $1 \leq i_0 < \dots < i_N \leq q$ , such that

$$\frac{Q_{i_0}(f)}{Q_{i_0}(g)} = \dots = \frac{Q_{i_N}(f)}{Q_{i_N}(g)}.$$

## N.B.

(i) Since the truncated level of the counting function in Theorem 4 is better, the number of hypersurfaces in Theorem 5 is much smaller than the previous results on unicity of meromorphic mappings sharing hypersurfaces (cf. [4], [5]).

(ii) In the case where  $d = 1$ , Theorem 5b) immediately gives us the following uniqueness theorem for meromorphic mappings into  $\mathbb{P}^n(\mathbb{C})$ , which may be linearly degenerate, sharing few hyperplanes in general position.

**Corollary 6.** *Let  $\{H_i\}_{i=1}^q$  be hyperplanes in  $\mathbb{P}^n(\mathbb{C})$  in general position. Let  $f$  and  $g$  be meromorphic mappings of  $\mathbb{C}^m$  into  $\mathbb{P}^n(\mathbb{C})$ . Assume that*

- (i)  $\dim(\text{Zero}H_i(f) \cap \text{Zero}H_i(g)) \leq m - 2$  for every  $1 \leq i < j \leq q$ ,
- (ii)  $f = g$  on  $\bigcup_{i=1}^q (\text{Zero}H_i(f) \cup \text{Zero}H_i(g))$ .

Let  $k$  be the dimension of the smallest linear subspace containing  $f(\mathbb{C}^m)$ . If  $q > 2(2n - k + 1)$  then  $f = g$ .

We may see that if  $f$  is linear nondegenerate, i.e.,  $k = n$ , then the condition of the above corollary is satisfied with  $q = 2n + 3$ . Therefore, Corollary 6 is a natural extension of the uniqueness for linear nondegenerate meromorphic mappings sharing  $2n + 3$  hyperplanes in  $\mathbb{P}^n(\mathbb{C})$  in general position given by Yan - Chen [3].

*Proof.* Let  $f = (f_0 : \cdots : f_n)$  and  $g = (g_0 : \cdots : g_n)$  be two reduced representations of  $f$  and  $g$  respectively. Let  $V(f)$  and  $V(g)$  be the smallest linear subspaces of  $\mathbb{P}^n(\mathbb{C})$  containing  $f(\mathbb{C}^m)$  and  $g(\mathbb{C}^m)$  respectively. It is easy to see that  $V(f)$  (resp.  $V(g)$ ) is the intersection of all hyperplanes which contain  $f(\mathbb{C}^m)$  (resp.  $g(\mathbb{C}^m)$ ). We may consider  $f$  (resp.  $g$ ) as a meromorphic mapping into  $V(f)$  (resp.  $V(g)$ ) which is nondegenerate over  $I_1(V(f))$  (resp.  $I_1(V(g))$ ). Of course,  $H_1, \dots, H_q$  are in  $n$ -subgeneral position with respect to both  $V(f)$  and  $V(g)$ .

Now let  $H$  be a hyperplane in  $\mathbb{P}^n(\mathbb{C})$  such that  $f(\mathbb{C}^m) \subset H$ . We denoted again by  $H$  the homogeneous linear form defining the hyperplane  $H$ . Suppose that  $g(\mathbb{C}^m) \not\subset H$ , i.e.,  $H(g) \not\equiv 0$ . Then we have  $H(g) = H(f) = 0$  on  $\bigcup_{i=1}^q \text{Zero}H_i(g)$ , and hence

$$\begin{aligned} T_g(r) &\geq N_{H(g)}(r) \geq \sum_{i=1}^q N_{H_i(g)}^{[1]}(r) + o(T_g(r)) \\ &\geq \frac{1}{H_{V(g)}(1) - 1} \sum_{i=1}^q N_{H_i(g)}^{[H_{V(g)}(1)-1]}(r) + o(T_g(r)) \\ &\geq \frac{1}{H_{V(g)}(1) - 1} (q - 2n + (H_{V(g)} - 1) - 1) T_g(r) + o(T_g(r)) \\ &\geq \frac{H_{V(g)} + 1}{H_{V(g)} - 1} T_g(r) + o(T_g(r)), \end{aligned}$$

(here, note that  $H_{V(g)}(1) - 1 = \dim V(g)$  and  $q \geq 2n + 3$ ). This is a contradiction. Therefore,  $g(\mathbb{C}^m) \subset H$ . This implies that  $g(\mathbb{C}^m) \subset V(f)$ , and hence  $V(g) \subset V(f)$ . Similarly, we have  $V(f) \subset V(g)$ . Then  $V(f) = V(g) = V$ .

We see that  $q > \frac{2(2n - k + 1)H_V(1)}{k + 1}$ , since  $H_V(1) = k + 1$ . Therefore, from Theorem 5 b), there exist  $n + 1$  hyperplanes  $H_{i_0}, \dots, H_{i_n}$ ,  $1 \leq i_0 < \cdots < i_n \leq q$  such that

$$\frac{H_{i_0}(f)}{H_{i_0}(g)} = \cdots = \frac{H_{i_n}(f)}{H_{i_n}(g)}.$$

This implies that  $f = g$ . □

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## 2. BASIC NOTIONS AND AUXILIARY RESULTS FROM NEVANLINNA THEORY

**2.1.** We set  $\|z\| = (|z_1|^2 + \cdots + |z_m|^2)^{1/2}$  for  $z = (z_1, \dots, z_m) \in \mathbb{C}^m$  and define

$$B(r) := \{z \in \mathbb{C}^m : \|z\| < r\}, \quad S(r) := \{z \in \mathbb{C}^m : \|z\| = r\} \quad (0 < r < \infty).$$

Define

$$v_{m-1}(z) := (dd^c \|z\|^2)^{m-1} \quad \text{and} \\ \sigma_m(z) := d^c \log \|z\|^2 \wedge (dd^c \log \|z\|^2)^{m-1} \text{ on } \mathbb{C}^m \setminus \{0\}.$$

For a divisor  $\nu$  on  $\mathbb{C}^m$  and for a positive integer  $M$  or  $M = \infty$ , define the counting function of  $\nu$  by

$$\nu^{[M]}(z) = \min \{M, \nu(z)\}, \\ n(t) = \begin{cases} \int_{|\nu| \cap B(t)} \nu(z) v_{m-1} & \text{if } m \geq 2, \\ \sum_{|z| \leq t} \nu(z) & \text{if } m = 1. \end{cases}$$

Similarly, we define  $n^{[M]}(t)$ .

Define

$$N(r, \nu) = \int_1^r \frac{n(t)}{t^{2m-1}} dt \quad (1 < r < \infty).$$

Similarly, define  $N(r, \nu^{[M]})$  and denote it by  $N^{[M]}(r, \nu)$ .

Let  $\varphi : \mathbb{C}^m \rightarrow \mathbb{C}$  be a meromorphic function. Denote by  $\nu_\varphi$  the zero divisor of  $\varphi$ . Define

$$N_\varphi(r) = N(r, \nu_\varphi), \quad N_\varphi^{[M]}(r) = N^{[M]}(r, \nu_\varphi).$$

For brevity, we will omit the character  $^{[M]}$  if  $M = \infty$ .

**2.2.** Let  $f : \mathbb{C}^m \rightarrow \mathbb{P}^n(\mathbb{C})$  be a meromorphic mapping. For arbitrarily fixed homogeneous coordinates  $(w_0 : \dots : w_n)$  on  $\mathbb{P}^n(\mathbb{C})$ , we take a reduced representation  $f = (f_0 : \dots : f_n)$ , which means that each  $f_i$  is a holomorphic function on  $\mathbb{C}^m$  and  $f(z) = (f_0(z) : \dots : f_n(z))$  outside the analytic subset  $\{f_0 = \dots = f_n = 0\}$  of codimension  $\geq 2$ . Set  $\|f\| = (|f_0|^2 + \dots + |f_n|^2)^{1/2}$ .

The characteristic function of  $f$  is defined by

$$T_f(r) = \int_{S(r)} \log \|f\| \sigma_m - \int_{S(1)} \log \|f\| \sigma_m.$$

**2.3.** Let  $\varphi$  be a nonzero meromorphic function on  $\mathbb{C}^m$ , which is occasionally regarded as a meromorphic map into  $\mathbb{P}^1(\mathbb{C})$ . The proximity function of  $\varphi$  is defined by

$$m(r, \varphi) = \int_{S(r)} \log \max(|\varphi|, 1) \sigma_m.$$

The Nevanlinna's characteristic function of  $\varphi$  is define as follows

$$T(r, \varphi) = N_{\frac{1}{\varphi}}(r) + m(r, \varphi).$$

Then

$$T_\varphi(r) = T(r, \varphi) + O(1).$$

The function  $\varphi$  is said to be small (with respect to  $f$ ) if  $\|T_\varphi(r) = o(T_f(r))$ . Here, by the notation “ $\| P$ ” we mean the assertion  $P$  holds for all  $r \in [0, \infty)$  excluding a Borel subset  $E$  of the interval  $[0, \infty)$  with  $\int_E dr < \infty$ .

**2.4. Lemma on logarithmic derivative** (see [13, Lemma 3.11]). *Let  $f$  be a nonzero meromorphic function on  $\mathbb{C}^m$ . Then*

$$\left\| m\left(r, \frac{\mathcal{D}^\alpha(f)}{f}\right) = O(\log^+ T(r, f)) \quad (\alpha \in \mathbb{Z}_+^m). \right.$$

Repeating the argument in [6, Proposition 4.5], we have the following.

**2.5. Proposition.** *Let  $\Phi_0, \dots, \Phi_k$  be meromorphic functions on  $\mathbb{C}^m$  such that  $\{\Phi_0, \dots, \Phi_k\}$  are linearly independent over  $\mathbb{C}$ . Then there exists an admissible set*

$$\{\alpha_i = (\alpha_{i1}, \dots, \alpha_{im})\}_{i=0}^k \subset \mathbb{Z}_+^m$$

with  $|\alpha_i| = \sum_{j=1}^m |\alpha_{ij}| \leq k$  ( $0 \leq i \leq k$ ) such that the following are satisfied:

(i)  $\{\mathcal{D}^{\alpha_i}\Phi_0, \dots, \mathcal{D}^{\alpha_i}\Phi_k\}_{i=0}^k$  is linearly independent over  $\mathcal{M}$ , i.e.,

$$\det(\mathcal{D}^{\alpha_i}\Phi_j) \neq 0.$$

(ii)  $\det(\mathcal{D}^{\alpha_i}(h\Phi_j)) = h^{k+1} \cdot \det(\mathcal{D}^{\alpha_i}\Phi_j)$  for any nonzero meromorphic function  $h$  on  $\mathbb{C}^m$ .

### 3. GENERALIZATION OF NOCHKA WEIGHTS

Let  $V$  be a complex projective subvariety of  $\mathbb{P}^n(\mathbb{C})$  of dimension  $k$  ( $k \leq n$ ). Let  $\{Q_i\}_{i=1}^q$  be  $q$  hypersurfaces in  $\mathbb{P}^n(\mathbb{C})$  of the common degree  $d$ , which are regarded as homogeneous polynomials in variables  $(x_0, \dots, x_n)$ . We regard  $I_d(V) = \frac{H_d}{I(V) \cap H_d}$  as a complex vector space. It is easy to see that

$$\text{rank}\{Q_i\}_{i \in R} \geq \dim V - \dim\left(\bigcap_{i \in R} Q_i \cap V\right).$$

Set  $\dim(\emptyset) = -1$ . Then, if  $\{Q_i\}_{i=1}^q$  is in  $N$ -subgeneral position, we have

$$\text{rank}\{Q_i\}_{i \in R} \geq \dim V - \dim\left(\bigcap_{i \in R} Q_i \cap V\right) = k + 1$$

for any subset  $R \subset \{1, \dots, q\}$  with  $\sharp R = N + 1$ .

Taking an  $\mathbb{C}$ -basis of  $I_d(V)$ , we may consider  $I_d(V)$  as a  $\mathbb{C}$ -vector space  $\mathbb{C}^M$  with  $M = H_V(d)$ .

Let  $\{H_i\}_{i=1}^q$  be  $q$  hyperplanes in  $\mathbb{C}^M$  passing through the coordinates origin. Assume that each  $H_i$  is defined by the linear equation

$$a_{i1}z_1 + \dots + a_{iM}z_M = 0,$$

where  $a_{ij} \in \mathbb{C}$  ( $j = 1, \dots, M$ ), not all zeros. We define the vector associated with  $H_i$  by

$$v_i = (a_{i1}, \dots, a_{iM}) \in \mathbb{C}^M.$$

For each subset  $R \subset \{1, \dots, q\}$ , the *rank* of  $\{H_i\}_{i \in R}$  is defined by

$$\text{rank}\{H_i\}_{i \in R} = \text{rank}\{v_i\}_{i \in R}.$$

Recall that the family  $\{H_i\}_{i=1}^q$  is said to be in  $N$ -subgeneral position if for any subset  $R \subset \{1, \dots, q\}$  with  $\sharp R = N + 1$ ,  $\bigcap_{i \in R} H_i = \{0\}$ , i.e.,  $\text{rank}\{H_i\}_{i \in R} = M$ .

By Lemmas 3.3 and 3.4 in [9], we have the following.

**Lemma 7.** *Let  $\{H_i\}_{i=1}^q$  be  $q$  hyperplanes in  $\mathbb{C}^{k+1}$  in  $N$ -subgeneral position, and assume that  $q > 2N - k + 1$ . Then there are positive rational constants  $\omega_i$  ( $1 \leq i \leq q$ ) satisfying the following:*

- i)  $0 < \omega_j \leq 1$ ,  $\forall i \in \{1, \dots, q\}$ ,
- ii) Setting  $\tilde{\omega} = \max_{j \in Q} \omega_j$ , one gets

$$\sum_{j=1}^q \omega_j = \tilde{\omega}(q - 2N + k - 1) + k + 1.$$

$$\text{iii)} \quad \frac{k+1}{2N-k+1} \leq \tilde{\omega} \leq \frac{k}{N}.$$

- iv) For  $R \subset Q$  with  $0 < \sharp R \leq N + 1$ , then  $\sum_{i \in R} \omega_i \leq \text{rank}\{H_i\}_{i \in R}$ .

v) Let  $E_i \geq 1$  ( $1 \leq i \leq q$ ) be arbitrarily given numbers. For  $R \subset Q$  with  $0 < \sharp R \leq N + 1$ , there is a subset  $R^o \subset R$  such that  $\sharp R^o = \text{rank}\{H_i\}_{i \in R^o} = \text{rank}\{H_i\}_{i \in R}$  and

$$\prod_{i \in R} E_i^{\omega_i} \leq \prod_{i \in R^o} E_i.$$

The above  $\omega_j$  are called *Nochka weights* and  $\tilde{\omega}$  is called *Nochka constant*.

**Lemma 8** (cf. [2, Lemma 3.2]). *Let  $H_1, \dots, H_q$  be  $q$  hyperplanes in  $\mathbb{C}^M$  ( $M \geq 2$ ), passing through the coordinates origin. Let  $k$  be a positive integer such that  $k \leq M$ . Then there exists a linear subspace  $L \subset \mathbb{C}^M$  of dimension  $k$  such that  $L \not\subset H_i$  ( $1 \leq i \leq q$ ) and*

$$\text{rank}\{H_{i_1} \cap L, \dots, H_{i_l} \cap L\} = \text{rank}\{H_{i_1}, \dots, H_{i_l}\}$$

for every  $1 \leq l \leq k, 1 \leq i_1 < \dots < i_l \leq q$ .

**Lemma 9.** *Let  $V$  be a complex projective subvariety of  $\mathbb{P}^n(\mathbb{C})$  of dimension  $k$  ( $k \leq n$ ). Let  $Q_1, \dots, Q_q$  be  $q$  ( $q > 2N - k + 1$ ) hypersurfaces in  $\mathbb{P}^n(\mathbb{C})$  in  $N$ -subgeneral position with respect to  $V$  of the common degree  $d$ . Then there are positive rational constants  $\omega_i$  ( $1 \leq i \leq q$ ) satisfying the following:*

- i)  $0 < \omega_i \leq 1$ ,  $\forall i \in \{1, \dots, q\}$ ,
- ii) Setting  $\tilde{\omega} = \max_{j \in Q} \omega_j$ , one gets

$$\sum_{j=1}^q \omega_j = \tilde{\omega}(q - 2N + k - 1) + k + 1.$$

$$\text{iii)} \quad \frac{k+1}{2N-k+1} \leq \tilde{\omega} \leq \frac{k}{N}.$$

- iv) For  $R \subset \{1, \dots, q\}$  with  $\sharp R = N + 1$ , then  $\sum_{i \in R} \omega_i \leq k + 1$ .

v) Let  $E_i \geq 1$  ( $1 \leq i \leq q$ ) be arbitrarily given numbers. For  $R \subset \{1, \dots, q\}$  with  $\sharp R = N + 1$ , there is a subset  $R^o \subset R$  such that  $\sharp R^o = \text{rank}\{Q_i\}_{i \in R^o} = k + 1$  and

$$\prod_{i \in R} E_i^{\omega_i} \leq \prod_{i \in R^o} E_i.$$

*Proof.* We assume that each  $Q_i$  is given by

$$\sum_{I \in \mathcal{I}_d} a_{iI} x^I = 0,$$

where  $\mathcal{I}_d = \{(i_0, \dots, i_n) \in \mathbb{N}_0^{n+1} : i_0 + \dots + i_n = d\}$ ,  $I = (i_0, \dots, i_n) \in \mathcal{I}_d$ ,  $x^I = x_0^{i_0} \dots x_n^{i_n}$  and  $a_{iI} \in \mathbb{C}$  ( $1 \leq i \leq q$ ,  $I \in \mathcal{I}_d$ ). Setting  $Q_i^*(x) = \sum_{I \in \mathcal{I}_d} a_{iI} x^I$ . Then  $Q_i^* \in H_d$ .

Taking a  $\mathbb{C}$ -basis of  $I_d(V)$ , we may identify  $I_d(V)$  with the  $\mathbb{C}$ -vector space  $\mathbb{C}^M$ , where  $M = H_V(d)$ . For each  $Q_i$ , denote by  $v_i$  the vector in  $\mathbb{C}^M$  which corresponds to  $[Q_i^*]$  by this identification. Denote by  $H_i$  the hyperplane in  $\mathbb{C}^M$  associated with the vector  $v_i$ .

Then for each arbitrary subset  $R \subset \{1, \dots, q\}$  with  $\sharp R = N + 1$ , we have

$$\dim\left(\bigcap_{i \in R} Q_i \cap V\right) \geq \dim V - \text{rank}\{[Q_i]\}_{i \in R} = k - \text{rank}\{H_i\}_{i \in R}.$$

Hence

$$\text{rank}\{H_i\}_{i \in R} \geq k - \dim\left(\bigcap_{i \in R} Q_i \cap V\right) \geq k - (-1) = k + 1.$$

By Lemma 8, there exists a linear subspace  $L \subset \mathbb{C}^M$  of dimension  $k + 1$  such that  $L \not\subset H_i$  ( $1 \leq i \leq q$ ) and

$$\text{rank}\{H_{i_1} \cap L, \dots, H_{i_l} \cap L\} = \text{rank}\{H_{i_1}, \dots, H_{i_l}\}$$

for every  $1 \leq l \leq k + 1$ ,  $1 \leq i_1 < \dots < i_l \leq q$ . Since  $\text{rank}\{H_i\}_{i \in R} \geq k + 1$ , it implies that for any subset  $R \subset \{1, \dots, q\}$  with  $\sharp R = N + 1$ , there exists a subset  $R' \subset R$  with  $\sharp R' = k + 1$  and  $\text{rank}\{H_i\}_{i \in R'} = k + 1$ . Hence, we get

$$\text{rank}\{H_i \cap L\}_{i \in R} \geq \text{rank}\{H_i \cap L\}_{i \in R'} = \text{rank}\{H_i\}_{i \in R'} = k + 1.$$

This yields that  $\text{rank}\{H_i \cap L\}_{i \in R} = k + 1$ , since  $\dim L = k + 1$ . Therefore,  $\{H_i \cap L\}_{i=1}^q$  is a family of  $q$  hyperplanes in  $L$  in  $N$ -subgeneral position.

By Lemma 7, there exist Nochka weights  $\{\omega_i\}_{i=1}^q$  for the family  $\{H_i \cap L\}_{i=1}^q$  in  $L$ . It is clear that assertions (i)-(iv) are automatically satisfied. Now for  $R \subset \{1, \dots, q\}$  with  $\sharp R = N + 1$ , by Lemma 7(v) we have

$$\sum_{i \in R} \omega_i \leq \text{rank}\{H_i \cap L\}_{i \in R} = k + 1$$

and there is a subset  $R^o \subset R$  such that:

$$\begin{aligned} \sharp R^o &= \text{rank}\{H_i \cap L\}_{i \in R^o} = \text{rank}\{H_i \cap L\}_{i \in R} = k + 1, \\ \prod_{i \in R} E_i^{\omega_i} &\leq \prod_{i \in R^o} E_i, \quad \forall E_i \geq 1 \quad (1 \leq i \leq q), \\ \text{rank}\{Q_i\}_{i \in R^o} &= \text{rank}\{H_i \cap L\}_{i \in R^o} = k + 1. \end{aligned}$$

Hence the assertion (v) is also satisfied. The lemma is proved.  $\square$



## 4. SECOND MAIN THEOREMS FOR HYPERSURFACES

Let  $\{Q_i\}_{i \in R}$  be a set of hypersurfaces in  $\mathbb{P}^n(\mathbb{C})$  of the common degree  $d$ . Assume that each  $Q_i$  is defined by

$$\sum_{I \in \mathcal{I}_d} a_{iI} x^I = 0,$$

where  $\mathcal{I}_d = \{(i_0, \dots, i_n) \in \mathbb{N}_0^{n+1} : i_0 + \dots + i_n = d\}$ ,  $I = (i_0, \dots, i_n) \in \mathcal{I}_d$ ,  $x^I = x_0^{i_0} \dots x_n^{i_n}$  and  $(x_0 : \dots : x_n)$  is homogeneous coordinates of  $\mathbb{P}^n(\mathbb{C})$ .

Let  $f : \mathbb{C}^m \longrightarrow V \subset \mathbb{P}^n(\mathbb{C})$  be an algebraically nondegenerate meromorphic mapping into  $V$  with a reduced representation  $f = (f_0 : \dots : f_n)$ . We define

$$Q_i(f) = \sum_{I \in \mathcal{I}_d} a_{iI} f^I,$$

where  $f^I = f_0^{i_0} \dots f_n^{i_n}$  for  $I = (i_0, \dots, i_n)$ . Then we see that  $f^*Q_i = \nu_{Q_i(f)}$  as divisors.

**Lemma 10.** *Let  $\{Q_i\}_{i \in R}$  be a set of hypersurfaces in  $\mathbb{P}^n(\mathbb{C})$  of the common degree  $d$  and let  $f$  be a meromorphic mapping of  $\mathbb{C}^m$  into  $\mathbb{P}^n(\mathbb{C})$ . Assume that  $\bigcap_{i \in R} Q_i \cap V = \emptyset$ . Then there exist positive constants  $\alpha$  and  $\beta$  such that*

$$\alpha \|f\|^d \leq \max_{i \in R} |Q_i(f)| \leq \beta \|f\|^d.$$

*Proof.* Let  $(x_0 : \dots : x_n)$  be homogeneous coordinates of  $\mathbb{P}^n(\mathbb{C})$ . Assume that each  $Q_i$  is defined by  $\sum_{I \in \mathcal{I}_d} a_{iI} x^I = 0$ .

Set  $Q_i(x) = \sum_{I \in \mathcal{I}_d} a_{iI} x^I$  and consider the following function

$$h(x) = \frac{\max_{i \in R} |Q_i(x)|}{\|x\|^d},$$

where  $\|x\| = (\sum_{i=0}^n |x_i|^2)^{\frac{1}{2}}$ .

Since the function  $h$  is positive continuous on  $V$ , by the compactness of  $V$ , there exist positive constants  $\alpha$  and  $\beta$  such that  $\alpha = \min_{x \in \mathbb{P}^n(\mathbb{C})} h(x)$  and  $\beta = \max_{x \in \mathbb{P}^n(\mathbb{C})} h(x)$ . Thus

$$\alpha \|f\|^d \leq \max_{i \in R} |Q_i(f)| \leq \beta \|f\|^d.$$

The lemma is proved. □

The following lemma is due to Lemma 4.2 in [2] with a slightly modification.

**Lemma 11** (cf. [2, Lemma 4.2]). *Let  $\{Q_i\}_{i=1}^q$  be a set of  $q$  hypersurfaces in  $\mathbb{P}^n(\mathbb{C})$  of the common degree  $d$ . Then there exist  $(H_V(d) - k - 1)$  hypersurfaces  $\{T_i\}_{i=1}^{H_V(d)-k-1}$  in  $\mathbb{P}^n(\mathbb{C})$  such that for any subset  $R \in \{1, \dots, q\}$  with  $\sharp R = \text{rank}\{Q_i\}_{i \in R} = k + 1$ , we get  $\text{rank}\{\{Q_i\}_{i \in R} \cup \{T_i\}_{i=1}^{M-k}\} = H_V(d)$ .*

*Proof.* For each  $R \subset \{1, \dots, q\}$  with  $\sharp R = \text{rank}\{Q_i\}_{i \in R} = k + 1$ , denote by  $V_R$  the set of all vectors  $v = (v_1, \dots, v_{H_V(d)-k-1}) \in (I_d(V))^{H_V(d)-k-1}$  such that  $\{\{Q_i\}_{i \in R}, v_1, \dots, v_{H_V(d)-k-1}\}$  is linearly dependent over  $\mathbb{C}$ . Then  $V_R$  is an algebraic subset of  $(I_d(V))^{H_V(d)-k-1}$ . Since  $\dim I_d(V) = H_V(d)$  and  $\text{rank}\{Q_i\}_{i \in R} = k + 1$ , there exists an element

$$v = (v_1, \dots, v_{H_V(d)-k-1}) \in (I_d(V))^{H_V(d)-k-1}$$

such that the family of vectors  $\{[Q_i]\}_{i \in R}, v_1, \dots, v_{H_V(d)-k-1}$  is linearly independent over  $\mathbb{C}$ , i.e.,  $v \notin V_R$ . Therefore  $V_R$  is a proper algebraic subset of  $(I_d(V))^{H_V(d)-k-1}$  for each  $R$ . This implies that

$$(I_d(V))^{H_V(d)-k-1} \setminus \bigcup_R V_R \neq \emptyset.$$

Hence, there is  $(T_1^+, \dots, T_{H_V(d)-k-1}^+) \in (I_d(V))^{H_V(d)-k-1} \setminus \bigcup_R V_R$ .

For each  $T_i^+$ , take a representation  $T_i \in H_d$  of  $T_i^+$ . Then

$$\text{rank}\{[Q_i]_{i \in R} \cup [T_i]_{i=1}^{H_V(d)-k-1}\} = \text{rank}\{[Q_i]_{i \in R} \cup [T_i]_{i=1}^{H_V(d)-k-1}\} = H_V(d)$$

for every subset  $R \in \{1, \dots, q\}$  with  $\sharp R = \text{rank}\{Q_i\}_{i \in R} = k+1$ .

The lemma is proved.  $\square$

#### Proof of Theorem 4.

We first prove the theorem in the case where all  $Q_i$  ( $i = 1, \dots, q$ ) do have the same degree  $d$ . It is easy to see that there is a positive constant  $\beta$  such that  $\beta \|f\|^d \geq |Q_i(f)|$  for every  $1 \leq i \leq q$ . Set  $Q := \{1, \dots, q\}$ . Let  $\{\omega_i\}_{i=1}^q$  be as in Lemma 9 for the family  $\{Q_i\}_{i=1}^q$ . Let  $\{T_i\}_{i=1}^{M-k}$  be  $(M-k)$  hypersurfaces in  $\mathbb{P}^n(\mathbb{C})$ , which satisfy Lemma 11.

Take a  $\mathbb{C}$ -basis  $\{[A_i]\}_{i=1}^{H_V(d)}$  of  $I_d(V)$ , where  $A_i \in H_d$ . Since  $f$  is nondegenerate over  $I_d(V)$ , it implies that  $\{A_i(f); 1 \leq i \leq H_V(d)\}$  is linearly independent over  $\mathbb{C}$ . Then there is an admissible set  $\{\alpha_1, \dots, \alpha_{H_V(d)}\} \subset \mathbb{Z}_+^m$  such that

$$W \equiv \det(\mathcal{D}^{\alpha_j} A_i(f)(1 \leq i \leq H_V(d)))_{1 \leq j \leq H_V(d)} \neq 0$$

and  $|\alpha_j| \leq H_V(d) - 1$  for all  $1 \leq j \leq H_V(d)$ .

For each  $R^o = \{r_1^0, \dots, r_{k+1}^0\} \subset \{1, \dots, q\}$  with  $\text{rank}\{Q_i\}_{i \in R^o} = \sharp R^o = k+1$ , set

$$W_{R^o} \equiv \det(\mathcal{D}^{\alpha_j} Q_{r_v^0}(f)(1 \leq v \leq k+1), \mathcal{D}^{\alpha_j} T_l(f)(1 \leq l \leq H_V(d) - k - 1))_{1 \leq j \leq H_V(d)}.$$

Since  $\text{rank}\{Q_{r_v^0}(1 \leq v \leq k+1), T_l(1 \leq l \leq H_V(d) - k - 1)\} = H_V(d)$ , there exists a nonzero constant  $C_{R^o}$  such that  $W_{R^o} = C_{R^o} \cdot W$ .

We denote by  $\mathcal{R}^o$  the family of all subsets  $R^o$  of  $\{1, \dots, q\}$  satisfying

$$\text{rank}\{Q_i\}_{i \in R^o} = \sharp R^o = k+1.$$

Let  $z$  be a fixed point. For each  $R \subset Q$  with  $\sharp R = N+1$ , we choose  $R^o \subset R$  such that  $R^o \in \mathcal{R}^o$  and  $R^o$  satisfies Lemma 9 v) with respect to numbers  $\{\frac{\beta \|f(z)\|^d}{|Q_i(f)(z)|}\}_{i=1}^q$ . On the other hand, there exists  $\bar{R} \subset Q$  with  $\sharp \bar{R} = N+1$  such that  $|Q_i(f)(z)| \leq |Q_j(f)(z)|, \forall i \in \bar{R}, j \notin \bar{R}$ . Since  $\bigcap_{i \in \bar{R}} Q_i = \emptyset$ , by Lemma 10, there exists a positive constant  $\alpha_{\bar{R}}$  such that

$$\alpha_{\bar{R}} \|f\|^d(z) \leq \max_{i \in \bar{R}} |Q_i(f)(z)|.$$

Then, we get

$$\begin{aligned} \frac{||f(z)||^{d(\sum_{i=1}^q \omega_i)} |W(z)|}{|Q_1^{\omega_1}(f)(z) \cdots Q_q^{\omega_q}(f)(z)|} &\leq \frac{|W(z)|}{\alpha_{\bar{R}}^{q-N-1} \beta^{N+1}} \prod_{i \in \bar{R}} \left( \frac{\beta ||f(z)||^d}{|Q_i(f)(z)|} \right)^{\omega_i} \\ &\leq A_{\bar{R}} \frac{|W(z)| \cdot ||f||^{d(k+1)}(z)}{\prod_{i \in \bar{R}^o} |Q_i(f)(z)|} \\ &\leq B_{\bar{R}} \frac{|W_{\bar{R}^o}(z)| \cdot ||f||^{dH_V(d)}(z)}{\prod_{i \in \bar{R}^o} |Q_i(f)(z)| \prod_{i=1}^{H_V(d)-k-1} |T_i(f)(z)|}, \end{aligned}$$

where  $A_{\bar{R}}, B_{\bar{R}}$  are positive constants.

Put  $S_{\bar{R}} = B_{\bar{R}} \frac{|W_{\bar{R}^o}|}{\prod_{i \in \bar{R}^o} |Q_i(f)| \prod_{i=1}^{H_V(d)-k-1} |T_i(f)|}$ . By the Lemma on logarithmic derivative, it is easy to see that

$$\int_{S(r)} \log^+ S_{\bar{R}}(z) \sigma_m = o(T_f(r)).$$

Therefore, for each  $z \in \mathbb{C}^m$ , we have

$$\log \left( \frac{||f(z)||^{d(\sum_{i=1}^q \omega_i)} |W(z)|}{|Q_1^{\omega_1}(f)(z) \cdots Q_q^{\omega_q}(f)(z)|} \right) \leq \log (||f||^{dH_V(d)}(z)) + \sum_{R \subset Q, \#R=N+1} \log^+ S_R.$$

Since  $\sum_{i=1}^q \omega_i = \tilde{\omega}_i(q - 2N + k - 1) + k + 1$  and by integrating both sides of the above inequality over  $S(r)$ , we have

(12)

$$\left\| d(q - 2N + k - 1 - \frac{H_V(d) - k - 1}{\tilde{\omega}}) T_f(r) \right\| \leq \sum_{i=1}^q \frac{\omega_i}{\tilde{\omega}} N_{Q_i(f)}(r) - \frac{1}{\tilde{\omega}} N_W(r) + o(T_f(r)).$$

**Claim.**  $\sum_{i=1}^q \omega_i N_{Q_i(f)}(r) - N_W(r) \leq \sum_{i=1}^q \omega_i N_{Q_i(f)}^{[H_V(d)-1]}(r)$ .

Indeed, let  $z$  be a zero of some  $Q_i(f)(z)$  and  $z \notin I(f) = \{f_0 = \cdots = f_n = 0\}$ . Since  $\{Q_i\}_{i=1}^q$  is in  $N$ -subgeneral position,  $z$  is not zero of more than  $N$  functions  $Q_i(f)$ . Without loss of generality, we may assume that  $z$  is zero of  $Q_i(f)$  for each  $1 \leq i \leq k \leq N$  and  $z$  is not zero of  $Q_i(f)$  for each  $i > N$ . Put  $R = \{1, \dots, N+1\}$ . Choose  $R^1 \subset R$  such that  $\#R^1 = \text{rank}\{Q_i\}_{i \in R^1} = k+1$  and  $R^1$  satisfies Lemma 9 v) with respect to numbers  $\{e^{\max\{\nu_{Q_i(f)}(z) - H_V(d) + 1, 0\}}\}_{i=1}^q$ . Then we have

$$\sum_{i \in R} \omega_i \max\{\nu_{Q_i(f)}(z) - H_V(d) + 1, 0\} \leq \sum_{i \in R^1} \max\{\nu_{Q_i(f)}(z) - H_V(d) + 1, 0\}.$$

This yields that

$$\begin{aligned} \nu_W(z) = \nu_{W_{R^1}}(z) &\geq \sum_{i \in R^1} \max\{\nu_{Q_i(f)}(z) - H_V(d) + 1, 0\} \\ &\geq \sum_{i \in R} \omega_i \max\{\nu_{Q_i(f)}(z) - H_V(d) + 1, 0\}. \end{aligned}$$

Hence

$$\begin{aligned}
\sum_{i=1}^q \omega_i \nu_{Q_i(f)}(z) - \nu_W(z) &= \sum_{i \in R} \omega_i \nu_{Q_i(f)}(z) - \nu_W(z) \\
&= \sum_{i \in R} \omega_i \min\{\nu_{Q_i(f)}(z), H_V(d) - 1\} \\
&\quad + \sum_{i \in R} \omega_i \max\{\nu_{Q_i(f)}(z) - H_V(d) + 1, 0\} - \nu_W(z) \\
&\leq \sum_{i \in R} \omega_i \min\{\nu_{Q_i(f)}(z), H_V(d) + 1\} \\
&= \sum_{i=1}^q \omega_i \min\{\nu_{Q_i(f)}(z), M\}.
\end{aligned}$$

Integrating both sides of this inequality, we get

$$\sum_{i=1}^q \omega_i N_{Q_i(f)}(r) - N_W(r) \leq \sum_{i=1}^q \omega_i N_{Q_i(f)}^{[H_V(d)-1]}(r).$$

This proves the claim.

Combining the claim and (12), we obtain

$$\begin{aligned}
&\| d(q - 2N + k - 1 - \frac{H_V(d) - k - 1}{\tilde{\omega}}) T_f(r) \\
&\leq \sum_{i=1}^q \frac{\omega_i}{\tilde{\omega}} N_{Q_i(f)}^{[H_V(d)-1]}(r) + o(T_f(r)) \\
&\leq \sum_{i=1}^q N_{Q_i(f)}^{[H_V(d)-1]}(r) + o(T_f(r)).
\end{aligned}$$

Since  $\tilde{\omega} \geq \frac{k+1}{2N-k+1}$ , the above inequality implies that

$$\left\| d \left( q - \frac{(2N - k + 1)H_V(d)}{k + 1} \right) T_f(r) \right\| \leq \sum_{i=1}^q N_{Q_i(f)}^{[H_V(d)-1]}(r) + o(T_f(r)).$$

Hence, the theorem is proved in the case where all  $Q_i$  do have the same degree.

We now prove the theorem in the general case where  $\deg Q_i = d_i$ . Applying the above case for  $f$  and the hypersurfaces  $Q_i^{\frac{d}{d_i}}$  ( $i = 1, \dots, q$ ) of the common degree  $d$ , we have

$$\begin{aligned} \left\| \left( q - \frac{(2N - k + 1)H_V(d)}{k + 1} \right) T_f(r) \right\| &\leq \frac{1}{d} \sum_{i=1}^q N_{Q_i^{\frac{d}{d_i}}(f)}^{[H_V(d)-1]}(r) + o(T_f(r)) \\ &\leq \sum_{i=1}^q \frac{1}{d} \frac{d}{d_i} N_{Q_i(f)}^{[H_V(d)-1]}(r) + o(T_f(r)) \\ &= \sum_{i=1}^q \frac{1}{d_i} N_{Q_i(f)}^{[H_V(d)-1]}(r) + o(T_f(r)). \end{aligned}$$

The theorem is proved.  $\square$

## 5. UNICITY OF MEROMORPHIC MAPPINGS SHARING HYPERSURFACES

**Lemma 13.** *Let  $f$  and  $g$  be nonconstant meromorphic mappings of  $\mathbb{C}^m$  into a complex projective subvariety  $V$  of  $\mathbb{P}^n(\mathbb{C})$ ,  $\dim V = k$  ( $k \leq n$ ). Let  $Q_i$  ( $i = 1, \dots, q$ ) be moving hypersurfaces in  $\mathbb{P}^n(\mathbb{C})$  in  $N$ -subgeneral position with respect to  $V$ ,  $\deg Q_i = d_i$ ,  $N \geq n$ . Put  $d = \text{lcm}(d_1, \dots, d_q)$  and  $M = \binom{n+d}{n} - 1$ . Assume that both  $f$  and  $g$  are nondegenerate over  $I_d(V)$ . Then  $\| T_f(r) = O(T_g(r))$  and  $\| T_g(r) = O(T_f(r))$  if  $q > \frac{(2N-k+1)H_V(d)}{k+1}$ .*

**Proof.** Using Theorem 4 for  $f$ , we have

$$\begin{aligned} \left\| \left( q - \frac{(2N - k + 1)H_V(d)}{k + 1} \right) T_f(r) \right\| &\leq \sum_{i=1}^q \frac{1}{d_i} N_{Q_i(f)}^{[H_V(d)-1]}(r) + o(T_f(r)) \\ &\leq \sum_{i=1}^q \frac{H_V(d) - 1}{d_i} N_{Q_i(f)}^{[1]}(r) + o(T_f(r)) \\ &= \sum_{i=1}^q \frac{H_V(d) - 1}{d_i} N_{Q_i(g)}^{[1]}(r) + o(T_f(r)) \\ &\leq q(H_V(d) - 1) T_g(r) + o(T_f(r)). \end{aligned}$$

Hence  $\| T_f(r) = O(T_g(r))$ . Similarly, we get  $\| T_g(r) = O(T_f(r))$ .

### Proof of Theorem 5.

Assume that  $f = (f_0 : \dots : f_n)$  and  $g = (g_0 : \dots : g_n)$  are reduced representations of  $f$  and  $g$ , respectively. Replacing  $Q_i$  by  $Q_i^{\frac{d}{d_i}}$  if necessary, without loss of generality, we may assume that  $d_i = d$  for all  $1 \leq i \leq q$ .

a) By Lemma 13, we have  $\| T_f(r) = O(T_g(r))$  and  $\| T_g(r) = O(T_f(r))$ . Suppose that  $f \neq g$ . Then there exist two indices  $s, t$  with  $0 \leq s < t \leq n$  such that  $H := f_s g_t - f_t g_s \not\equiv 0$ . By the assumption (ii) of the theorem, we have  $H = 0$  on  $\bigcup_{i=1}^q (\text{Zero } Q_i(f) \cup \text{Zero } Q_i(g))$ .

Therefore, we have

$$\nu_H^0 \geq \sum_{i=1}^q \min\{1, \nu_{Q_i(f)}^0\}$$

outside an analytic subset of codimension at least two. This follows that

$$(14) \quad N_H(r) \geq \sum_{i=1}^q N_{Q_i(f)}^{[1]}(r).$$

On the other hand, by the definition of the characteristic function and by the Jensen formula, we have

$$\begin{aligned} N_H(r) &= \int_{S(r)} \log |f_s g_t - f_t g_s| \sigma_m \\ &\leq \int_{S(r)} \log \|f\| \sigma_m + \int_{S(r)} \log \|g\| \sigma_m \\ &= T_f(r) + T_g(r). \end{aligned}$$

Combining this and (14), we obtain

$$T_f(r) + T_g(r) \geq \sum_{i=1}^q N_{Q_i(f)}^{[1]}(r).$$

Similarly, we have

$$T_f(r) + T_g(r) \geq \sum_{i=1}^q N_{Q_i(g)}^{[1]}(r).$$

Summing-up both sides of the above two inequalities, we have

$$(15) \quad 2(T_f(r) + T_g(r)) \geq \sum_{i=1}^q N_{Q_i(f)}^{[1]}(r) + \sum_{i=1}^q N_{Q_i(g)}^{[1]}(r).$$

From (15) and applying Theorem 4 for  $f$  and  $g$ , we have

$$\begin{aligned} &2(T_f(r) + T_g(r)) \\ &\geq \sum_{i=1}^q \frac{1}{H_V(d) - 1} N_{Q_i(f)}^{[H_V(d)-1]}(r) + \sum_{i=1}^q \frac{1}{H_V(d) - 1} N_{Q_i(g)}^{[H_V(d)-1]}(r) \\ &\geq \frac{d}{H_V(d) - 1} \left( q - \frac{(2N - k + 1)H_V(d)}{k + 1} \right) (T_f(r) + T_g(r)) + o(T_f(r) + T_g(r)). \end{aligned}$$

Letting  $r \rightarrow +\infty$ , we get

$$\begin{aligned} 2 &\geq \frac{d}{H_V(d) - 1} \left( q - \frac{(2N - k + 1)H_V(d)}{k + 1} \right), \\ \text{i.e., } q &\leq \frac{2(H_V(d) - 1)}{d} + \frac{(2N - k + 1)H_V(d)}{k + 1}. \end{aligned}$$

This is a contradiction. Hence  $f = g$ . The assertion a) is proved.

b) Again, by Lemma 13, we have  $\|T_f(r) = O(T_g(r))$  and  $\|T_g(r) = O(T_f(r))$ . Suppose that the assertion b) of the theorem does not hold.

By changing indices if necessary, we may assume that

$$\underbrace{\frac{Q_1(f)}{Q_1(g)} \equiv \dots \equiv \frac{Q_{k_1}(f)}{Q_{k_1}(g)}}_{\text{group 1}} \not\equiv \underbrace{\frac{Q_{k_1+1}(f)}{Q_{k_1+1}(g)} \equiv \dots \equiv \frac{Q_{k_2}(f)}{Q_{k_2}(g)}}_{\text{group 2}},$$

$$\not\equiv \underbrace{\frac{Q_{k_2+1}(f)}{Q_{k_2+1}(g)} \equiv \dots \equiv \frac{Q_{k_3}(f)}{Q_{k_3}(g)}}_{\text{group 3}} \not\equiv \dots \not\equiv \underbrace{\frac{Q_{k_{s-1}+1}(f)}{Q_{k_{s-1}+1}(g)} \equiv \dots \equiv \frac{Q_{k_s}(f)}{Q_{k_s}(g)}}_{\text{group s}},$$

where  $k_s = q$ .

Since the assertion b) of the theorem does not hold, the number of elements of each group is at most  $N$ . For each  $1 \leq i \leq q$ , we set

$$\sigma(i) = \begin{cases} i + N & \text{if } i + N \leq q, \\ i + N - q & \text{if } i + N > q \end{cases}$$

and

$$P_i = Q_i(f)Q_{\sigma(i)}(g) - Q_i(g)Q_{\sigma(i)}(f).$$

Then  $\frac{Q_i(f)}{Q_i(g)}$  and  $\frac{Q_{\sigma(i)}(f)}{Q_{\sigma(i)}(g)}$  belong to two distinct groups, and hence  $P_i \neq 0$  for every  $1 \leq i \leq q$ . It is easy to see that

$$\begin{aligned} \nu_{P_i}(z) &\geq \min\{\nu_{Q_i(f)}(z), \nu_{Q_i(g)}(z)\} + \min\{\nu_{Q_{\sigma(i)}(f)}(z), \nu_{Q_{\sigma(i)}(g)}(z)\} \\ &\quad + \sum_{\substack{j=1 \\ j \neq i, \sigma(i)}}^q \min\{\nu_{Q_j(f)}(z), 1\} \\ &\geq \sum_{j=i, \sigma(i)} \left( \min\{\nu_{Q_j(f)}(z), H_V(d) - 1\} + \min\{\nu_{Q_j(g)}(z), H_V(d) - 1\} \right. \\ &\quad \left. - (H_V(d) - 1) \min\{\nu_{Q_j(f)}(z), 1\} \right) + \sum_{\substack{j=1 \\ j \neq i, \sigma(i)}}^q \min\{\nu_{Q_j(f)}(z), 1\}. \end{aligned}$$

for all  $z$  in  $\mathbb{C}^m$ .

Integrating both sides of this inequality, we get

$$\begin{aligned} (16) \quad || N_{P_i}(r) &\geq \sum_{j=i, \sigma(i)} \left( N_{Q_j(f)}^{[H_V(d)-1]}(r) + N_{Q_j(g)}^{[H_V(d)-1]}(r) - (H_V(d) - 1) N_{Q_j(f)}^{[1]}(r) \right) \\ &\quad + \sum_{\substack{j=1 \\ j \neq i, \sigma(i)}}^q N_{Q_j(f)}^{[1]}(r). \end{aligned}$$

Repeating the same argument as in the proof of Theorem 5, by Jensen's formula and by the definition of the characteristic function, we have

$$(17) \quad || N_{P_i}(r) \leq d(T_f(r) + T_g(r))$$

From (16) and (17), we get

$$\begin{aligned} || d(T_f(r) + T_g(r)) &\geq \sum_{j=i, \sigma(i)} \left( N_{Q_j(f)}^{[H_V(d)-1]}(r) + N_{Q_j(g)}^{[H_V(d)-1]}(r) - (H_V(d) - 1) N_{Q_j(f)}^{[1]}(r) \right) \\ &\quad + \sum_{\substack{j=1 \\ j \neq i, \sigma(i)}}^q N_{Q_j(f)}^{[1]}(r). \end{aligned}$$

Summing-up both sides of this inequality over all  $1 \leq i \leq q$ , we obtain

$$\begin{aligned} || dq(T_f(r) + T_g(r)) &\geq 2 \sum_{j=1}^q \left( N_{Q_j(f)}^{[H_V(d)-1]}(r) + N_{Q_j(g)}^{[H_V(d)-1]}(r) \right) + (q - 2H_V(d)) \sum_{j=1}^q N_{Q_j(f)}^{[1]}(r) \\ &\geq 2d \left( q - \frac{(2N - k + 1)H_V(d)}{k + 1} \right) (T_f(r) + T_g(r)) + o(T_f(r)). \end{aligned}$$

Letting  $r \rightarrow +\infty$ , we get

$$\begin{aligned} dq &\geq 2d \left( q - \frac{(2N - k + 1)H_V(d)}{k + 1} \right), \\ \text{i.e., } q &\leq \frac{2(2N - k + 1)H_V(d)}{k + 1}. \end{aligned}$$

This is a contradiction.

Hence the assertion b) holds. The theorem is proved.  $\square$

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